University of California, Berkeley
Physics H7A Fall 1998 (*Strovink*)

SOLUTION TO PROBLEM SET 9

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1.

(a.) The Taylor series for $\ln(1-x)$ is found as follows:

$$f(x) = \ln(1-x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$$

where $f^{(n)}$ denotes the nth derivative of f.

$$\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \cdots$$
$$\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

We do the same for $f(x) = 1/(1+x)$.

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$$

(b.) We have two functions $c(x)$ and $s(x)$ related as follows:

$$\frac{ds}{dx} = c \qquad \frac{dc}{dx} = s$$

An easy way to approach this problem is to solve these differential equations simultaneously. However, as is the case for most "easy" ways to do things, the mathematics leading up to the solution is somewhat advanced. Instead we will use the Taylor series to solve it. Expanding around $x = 0$,

$$s(x) = s(0) + c(0)x + \frac{1}{2}s(0)x^2 + \cdots$$

$$c(x) = c(0) + s(0)x + \frac{1}{2}c(0)x^2 + \cdots$$

Adding these two equations, we see that

$$s(x) + c(x) = [s(0) + c(0)] \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

We recognize the sum as the Taylor series of e^x.

$$s(x) + c(x) = [s(0) + c(0)]e^x$$

2.

(a.) French problem 1-4(b).
The magnitude of a complex number $a + ib$, where a and b are real, is just $\sqrt{a^2+b^2}$. The phase angle θ is equal to $\tan^{-1}(b/a)$, where the quadrant is determined by the signs of both a and b. The first vector $(2 + i\sqrt{3})$ has length $\sqrt{7}$ and phase $\theta = \tan^{-1}(\sqrt{3}/2) = 40.9°$. The second vector $(2 - i\sqrt{3})^2$ is merely the square of the complex conjugate of the first vector. Therefore it has length 7 and $-2\times$ the phase, or $-81.8°$.

(b.) French problem 1-9.
The value of i^i is a little odd, but here goes. We need to know how to find the log of a complex number:

$$\ln z = \ln |z|e^{i\theta} = \ln |z| + i\theta$$

There is an ambiguity here, because we can always add an integer multiple of 2π to θ. Here we will choose not to do so, but simply take the value of θ to be between $-\pi$ and π. Thus we obtain

$$i^i = e^{i \ln i} = e^{i^2\pi/2} = e^{-\pi/2} = 0.2079$$

This means that paying 20 cents is bargain, but a very small one.

(c.) Prove $(\cos\theta + i\sin\theta)^n = (\cos n\theta + i\sin n\theta)$. This is pretty trivial when we remember

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta}$$
$$= \cos n\theta + i\sin n\theta$$

3. French problem 3-15.
An oscillatory system loses energy according

to $E = E_0 e^{-\gamma t}$. We define the Q value as $Q \equiv \omega_0/\gamma$.

(a.) Middle C on a piano is played, and the energy decreases to half of its initial value in one second. The frequency is 256 Hz. The angular frequency is this times 2π, so ω_0 =1608.5/sec. We find γ from

$$\frac{1}{2} = (e^{-\gamma/2})^2 \;\Rightarrow\; \gamma = 0.693$$

Lastly , the Q of the oscillator is

$$Q = 1608.5/0.693 = 2321$$

(b.) The note one octave above is struck (512 Hz). The decay time is the same, so the Q value is simply doubled: Q=4642.

(c.) A damped harmonic oscillator has mass $m = 0.1$ kg, spring constant $k = 0.9$ N/m, and a damping constant b. The energy decays to $1/e$ in 4 seconds. This means that

$$\frac{1}{e} = e^{-4\gamma} \;\Rightarrow\; \gamma = 0.25 \text{ sec}^{-1}$$
$$\Rightarrow\; b = m\gamma = 0.025 \text{ kg sec}^{-1}$$

The natural frequency $\omega_0 = \sqrt{k/m} = 3$ Hz. Finally, the Q of the oscillator is $Q = \omega_0/\gamma = 12$.

4. At $t = 0$, the bullet collides inelastically with the block, so only the momentum is conserved. The final velocity of the block and bullet is given by

$$mv_0 = (M+m)v \;\Rightarrow\; v = \frac{mv_0}{M+m}$$

We now have the initial conditions for the oscillation. The initial position is $x(0) = 0$ and the initial velocity is $v(0) = v$. The frequency ω is given as usual by $\sqrt{k/\text{mass}}$, but the mass in question is the total mass of the system:

$$\omega = \sqrt{\frac{k}{M+m}}$$

The solution is given by $x(t) = \Re[A\exp(i(\omega t + \phi))]$. The initial position tells us that $\cos\phi = 0$. This is ambiguous because the cosine is zero in two places in one oscillation. We want a place where it is zero and rising, because we know that x is increasing at the instant of contact. This is

$$\phi = -\pi/2$$

The solution is now

$$x(t) = \Re[A\exp(i(\omega t - \pi/2)])$$

We differentiate to get the velocity, and evaluate this at $t = 0$.

$$v(0) = v = -A\omega\sin(-\pi/2) = A\omega \;\Rightarrow\; A = \frac{v}{\omega}$$

This gives the final result for the amplitude

$$A = \frac{mv_0}{\sqrt{k(M+m)}}$$

5. French 4-5.

(a.) A pendulum is forced by moving the point of support. The coordinate x gives the location of the pendulum bob, and ξ gives the location of the point of support. The forces on the pendulum are the damping force, which we must assume to be proportional to the absolute velocity of the pendulum, and the force of gravity. The force of gravity depends on the angle by which the pendulum is raised. This is proportional to the distance that the pendulum bob is displaced from the point of support, $x - \xi$. This gives the equation of motion

$$m\frac{d^2x}{d^t2} = -b\frac{dx}{dt} - \frac{mg}{l}(x - \xi)$$

Using $\omega_0^2 = g/l$ and $\gamma = b/m$, we put this in the standard form with ξ as a forcing term.

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \omega_0^2\xi$$

(b.) The motion of the point of support is given by $\xi(t) = \xi_0\cos\omega t$. We use the formula for the amplitude of forced oscillation, but we note that in this case, the equation is

$$\frac{d^2x}{dt^2} + \gamma\frac{dx}{dt} + \omega_0^2 x = \omega_0^2\xi_0\cos\omega t$$

The constant $\omega_0^2\xi_0$ takes the place of the F_0/m we normally see in this type of equation. The amplitude of the oscillation is thus given by

$$A(\omega) = \frac{\omega_0^2\xi_0}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}}$$

At exact resonance, $\omega = \omega_0$ and the amplitude is

$$A(\omega_0) = \xi_0\frac{\omega_0}{\gamma} = Q\xi_0$$

Now we want to find Q. We are given that the forcing amplitude is $\xi_0 = 1$ mm. The length of the pendulum is $l = 1$ m, so this gives $\omega_0 = 3.13$ sec^{-1}. We know that the amplitude falls off by a factor of e after 50 swings, or 50 periods. We know that $A = A_0\exp(-\gamma t/2)$ so $\gamma t = 2$. t is 50 periods, or $100\pi/\Omega$, where Ω is the frequency of free oscillation

$$\Omega = \sqrt{\omega_0^2+\gamma^2/4}$$

Then

$$\begin{aligned}\gamma t &= 2\\ \gamma\frac{100\pi}{\Omega} &= 2\\ 50\pi\gamma &= \sqrt{\omega_0^2+\gamma^2/4}\\ (50\pi)^2\gamma^2 &= \omega_0^2+\gamma^2/4\end{aligned}$$

Plugging in the numbers, we get $\gamma = 0.0199$. This gives us $Q = 157$, and $A = 15.7$ cm.

(c.) We want to find the frequencies where the amplitude is half of the resonant value. We merely solve

$$\frac{\omega_0^2\xi_0}{\sqrt{(\omega_0^2-\omega^2)^2+\gamma^2\omega^2}} = \xi_0\frac{\omega_0}{2\gamma}$$

This gives

$$4\omega_0^2\gamma^2 = (\omega_0^2-\omega^2)^2+\gamma^2\omega^2$$

Turning this into a quadratic equation for ω^2, we get

$$0 = \omega^4 + (\gamma^2-2\omega_0^2)\omega^2+\omega_0^4-4\omega_0^2\gamma^2$$

The solutions to this are

$$\omega^2 = \frac{1}{2}\left(2\omega_0^2-\gamma^2\pm\sqrt{\gamma^4+12\gamma^2\omega_0^2}\right)$$

Plugging in the numbers, the two frequencies are $\omega = 3.147$ sec^{-1} and $\omega = 3.113$ sec^{-1}.

6. French 4-8.

(a.) A mass is under the influence of a viscous force $F = -bv$. Let $\gamma = b/m$ as usual. The equation of motion is

$$\frac{dv}{dt}+\gamma v = 0$$

We can easily solve this equation by direct integration.

$$v(t) = v_0 e^{-\gamma t}$$

We simply integrate this equation with respect to t to get the position.

$$x(t) = C - \frac{v_0}{\gamma}e^{-\gamma t}$$

C is the integration constant that will allow us to fit an initial condition.

(b.) A driving force $F = F_0\cos\omega t$ is turned on. We want to find the steady state motion. We will use a complex exponential for the forcing term, with the understanding that we take the real part when we're done. The new equation of motion is

$$\frac{d^2x}{dt^2}+\gamma\frac{dx}{dt} = \frac{F_0}{m}e^{i\omega t}$$

Assume a solution of the form

$$x(t) = Ae^{i\omega t}$$

where A is a complex number. Plugging this into the equation of motion, we see that

$$-\omega^2 A + i\omega\gamma A = \frac{F_0}{m} \;\Rightarrow\; A = \frac{F_0/m}{-\omega^2+i\omega\gamma}$$

We write the denominator as the product of a magnitude and a phase. The magnitude is $\sqrt{\omega^4+\omega^2\gamma^2}$. The denominator has a negative real part and a positive imaginary part, so it is in the second quadrant with phase $\pi - \arctan(\gamma/\omega)$. Since the numerator is real, the phase of A is minus the phase of its denominator, or $\arctan(\gamma/\omega) - \pi$. According to the notation of the problem, the phase of the oscillation is $-\delta$, so we find

$$\delta = \pi - \arctan(\gamma/\omega)$$

The amplitude of the oscillation is just the magnitude of A, given by

$$|A| = \frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}$$

The general solution to the problem is

$$x(t) = C - \frac{v_0}{\gamma}e^{-\gamma t} - \frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}\cos(\omega t + \tan^{-1}(\gamma/\omega))$$

At $t = 0$, we want $x = 0$. At $t = 0$, the last term B in the general solution is

$$\begin{aligned} B(0) &= -\frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}\cos(\tan^{-1}(\gamma/\omega)) \\ &= -\frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}\frac{\omega}{\sqrt{\omega^2 + \gamma^2}} \\ &= -\frac{F_0/m}{\omega^2 + \gamma^2} \end{aligned}$$

Thus the condition $x(0) = 0$ gives us one equation for C and v_0:

$$C = \frac{v_0}{\gamma} + \frac{F_0/m}{\omega^2 + \gamma^2}$$

The first time derivative of B, evaluated at $t = 0$, is

$$\begin{aligned} \dot{B}(0) &= \frac{F_0/m}{\sqrt{\omega^2 + \gamma^2}}\sin(\tan^{-1}(\gamma/\omega)) \\ &= \frac{F_0\gamma/m}{\omega^2 + \gamma^2} \end{aligned}$$

Then, requiring the first time derivative of the general solution to vanish at $t = 0$, the second equation for v_0 and C is

$$\begin{aligned} 0 &= 0 + v_0 + \frac{F_0\gamma/m}{\omega^2 + \gamma^2} \\ v_0 &= -\frac{F_0\gamma/m}{\omega^2 + \gamma^2} \end{aligned}$$

Plugging this value of v_0 into the first equation,

$$C = -\frac{F_0/m}{\omega^2 + \gamma^2} + \frac{F_0/m}{\omega^2 + \gamma^2} = 0$$

Collecting these results, the solution satisfying both boundary conditions is

$$x(t) = \frac{F_0/m}{\omega^2 + \gamma^2}e^{-\gamma t} - \frac{F_0/m}{\sqrt{\omega^4 + \omega^2\gamma^2}}\cos(\omega t - \tan^{-1}(\gamma/\omega))$$

7. Middle C is 256 Hz, and C above it is double that frequency, or 512 Hz. The scale is divided into 6 whole steps, or 12 half steps. The note after each half step is a constant multiple f of the frequency of the previous note. When we have gone up twelve half steps, the frequency will have doubled. The constant factor f is thus given by $f^{12} = 2$ or $f = 2^{1/12}$.

(a.) The frequencies in the scale are thus C=256, D=287.4, E=322.5, F=341.7, G=383.6, A=430.5, B=483.3, C=512 Hz. These are zero, two, four, five, seven, nine, eleven, and twelve half steps above middle C, respectively.

(b.) Middle C's third harmonic is three times its fundamental frequency, or 768 Hz. The second harmonic of G is 767.133 Hz. The beat frequency is always just the difference between the two frequencies. In this case the beat frequency is 0.867 Hz, which is easily audible to the piano tuner.

8. An undriven oscillator that is underdamped has a Q of 100. We want to know how many oscillations it takes to damp by a factor of e^{π}. This just means that $\gamma t/2 = \pi$. Now we want to write t in terms of the number of oscillations n. This takes n times the period, or $t = 2\pi n/\omega_0$. This gives $\gamma n/\omega_0 = 1$. Remember that $Q = \omega_0/\gamma$, so $n = Q = 100$.